Jesse Gerald Chaney

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UNITED STATES
NAVAL POSTGRADUATE SCHOOL

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ABSTRACT

The theory is postulated that the variational technique for discovering the laws governing a physical problem, formulated within a complex domain, is primarily a result of requiring analyticity in the variational formulation. In support of the theory, both in low frequency network theory and in antenna theory, examples are shown in which the governing circuit laws are obtainable from functions which do not possess the stationary property simply by requiring analyticity in the functions. The resulting circuitry is the same as that found from the conventional variational formulation. In consequence of the examples, for any number of trial functions used in setting up the currents along any number of antennas, it becomes evident that the current modes obey the laws of ordinary circuit analysis. The generalized circuitry is thereby extended to any thin wire antenna configuration.

INTRODUCTION

There seems to be an increasing trend in applying variational techniques to the solution of problems formulated within a complex domain. Basically, a complex parameter is introduced into a function as a control of the perturbation of some quantity. The function is then optimized by requiring the vanishing of the first term of a Taylor's expansion about some value of the parameter. The perturbation thus determined is customarily considered to be the best solution for the trial functions used, provided the original function possesses the stationary property, that is, provided the original function also has a vanishing first term in its Taylor's expansion about the zero value of the parameter.

Rumsey[1] states that there is no assurance that the best approximation to the true value of a quantity is determined by the variational technique if the assumed distribution about the true distribution is completely arbitrary. He[2] further states that his reaction theorem produces the best value in the sense that an approximate source looks like a true source to any source which can be used as an observation. The latter statement implies that all approximate sources are required to obey the same physical laws which apply to the true sources. Thus, it seems likely that the best results obtained by the variational principle are those in which the governing physical laws are discovered and/or imposed.

1. V. H. Rumsey, "Reaction Concept in Electromagnetic Theory", Phys. Rev., Vol. 94, pp. 1483-1491; June, 1954, p. 1485.
2. Ibid.

The existence of a Taylor's expansion requires the function to be analytic in the parameters. This requirement is a very stringent one. The real and imaginary components are not independent.[3] Indeed, the requirement of analyticity leads to the same solution of a physical problem as that obtained by the conventional stationary requirement.

Hereafter, the word <u>optimize</u> will be used in the sense of discovering the physical laws governing a given problem. Several examples, both in low frequency circuit networks and in antenna networks, will be solved to illustrate the equivalence of the aforementioned two methods of optimizing variational expressions. In this sense of equivalence, a variational expression need not necessarily possess the stationary property.

In the examples, the variational functions which are essentially quadric forms in the parameters may be optimized by requiring analyticity through the existence of vanishing first derivatives, whereas, the functions which are essentially bilinear forms in the parameters and their complex conjugates, may be optimized by means of requiring analyticity through the vanishing of the partial derivatives with respect to the conjugates of the parameters.

NOTATION

For convenience, the shorthand notation of tensor analysis[4] will be used. That is, the summation symbols will be dropped and summations will be indicated by means of a combination of superscripts

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3. P. M. Morse, and H. Feschbach, "Methods of Theoretical Physics", McGraw Hill Book Co., New York, N. Y., 1953, p. 370.

4. Gabriel Kron, "Tensor Analysis of Networks", John Wiley and Sons, New York, N. Y., 1939.

and subscripts. The occurrence of a lower case index as both a superscript and as a subscript implies summation over the range of the index, unless the index is enclosed within a parenthesis. A capital index is used to represent a single value of the range. Thus,

$$\sum_{r=1}^{n}\sum_{s=1}^{n} Z_{(r)(s)}\, I^{(r)} I^{(s)} \equiv Z_{rs}\, I^{r} I^{s}, \quad r, s = 1, \cdots, n,$$

and $\qquad Z_{RS}\, I^{R} I^{S} \qquad$ is only a single term.

The Kronecker delta,

$$\delta^{i}_{j} = 1,\ i = j;\quad \delta^{i}_{j} = 0,\ i \neq j,$$

will be used. Also, the impulse function,

$$\delta(s) = 0,\ s \neq 0;\quad \delta(s) = \infty,\ s = 0;$$

$$\int_{-\infty}^{\infty} \delta(s)\, ds = 1;$$

will be used. The latter should not be confused with δI which is the first variation of I. The conventional exp $(j\omega t)$ phasor notation will be implied.

LUMPED NETWORKS

Some simple examples of lumped network theory will be used introductorily.

Consider the network defined by,

$$Z_{ij}\, I^{j} = V_{i},\quad i, j = 1, \cdots, n. \tag{1}$$

Then, either the quadric form,

$$W = \tfrac{1}{2} Z_{ij}\, I^{i} I^{j}, \tag{2}$$

or the bilinear form,

$$\tilde{W} = \tfrac{1}{2}\, z_{ij}\, \bar{I}^{i} I^{j}, \tag{3}$$

may be written. In the bilinear form, $\bar{I}^{i}$ is the complex conjugate of I^{i}.

The real part of W is the time instantaneous power less the time average power loss per cycle in the network [5]. The real part of $\tilde{W}$ is the time average power loss and the imaginary part of $\tilde{W}$ is the peak reactive power of the circuit. Each has its on sphere of usefulness.

Given the quadric form (2), upon assuming the impedance matrix, $\| z_{ij} \|$, to be symmetric in the subscripts, the voltage laws for the network may be found by,

$$\frac{\partial W}{\partial I^{i}} = z_{ij} I^{j} = V_{i} \,. \tag{4}$$

Likewise, for the bilinear form (3),

$$2\, \frac{\partial \tilde{W}}{\partial \bar{I}^{i}} = z_{ij}\, I^{j} = V_{i} \,, \tag{5}$$

with $\| z_{ij} \|$ not necessarily being symmetric in the subscripts, unless the reciprocity theorem is implied.

If, in equation (1), $V_{i} = V_{(k)}\, \delta_{i}^{(k)}$, the network becomes a two terminal one. The input impedance may be written,

$$z_{k} = \frac{z_{ij}\, I^{i} I^{j}}{(I^{k})^{2}} \,. \tag{6}$$

Suppose it is desired to find the error produced in z_{k} if the I^{i} are replaced by approximate values, $I^{i} + \epsilon \eta^{i}$, ϵ being a parameter for controling the relative size of the approximations for the currents.

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5. Rumsey, op. cit., p. 1484.

The impedance is written as,

$$Z_K(\epsilon) = \frac{Z_{ij}\,(I^i + \epsilon\eta^i)(I^j + \epsilon\eta^j)}{(I^K + \epsilon\eta^K)^2} \quad . \tag{7}$$

Since $Z_K(0) = Z_K$, the error may be determined by expanding $Z_K(\epsilon)$ in a Taylor's series about $\epsilon = 0$.

The existence of a Taylor's series for a function of a complex variable requires the function to be analytic. One way of assuring analyticity is to specify the first derivative, for if the first derivative exists, so do all order derivatives [6]. Now,

$$\frac{d}{d\epsilon} Z_K(\epsilon) = \frac{2}{I^K + \epsilon\eta^K}\left[\frac{Z_{ij}\eta^i(I^j + \epsilon\eta^j)}{I^K + \epsilon\eta^K} - \eta^{(K)} Z_{(K)}(\epsilon)\right], \tag{8}$$

or

$$\frac{d}{d\epsilon} Z_K(0) = \frac{2}{I^K}\left[\frac{Z_{ij}\eta^i I^j}{I^K} - \eta^{(K)} Z_{(K)}\right] \quad . \tag{9}$$

Inspection of (9) reveals that perhaps the simplest solution is to require the first derivative to vanish. Further inspection reveals that since,

$$\frac{V_{(K)}\,\delta_i^{(K)}\eta^i}{I^{(K)}} = \frac{V_{(K)}\,\eta^{(K)}}{I^{(K)}} = Z_{(K)}\,\eta^{(K)},$$

the derivative will vanish provided,

$$Z_{ij}\,I^j = V_K\,\delta_i^K \quad . \tag{10}$$

Consequently, the voltage law for the network has been determined by requiring analyticity at $\epsilon = 0$.

Proceeding, the nth derivative is,

$$\frac{d^n}{d\epsilon^n} Z_K(0) = (n-1)\,n!\left(-\frac{\eta^{(K)}}{I^{(K)}}\right)^n\left[\frac{Z_{ij}\,\eta^i\eta^j}{(\eta^{(K)})^2} - Z_K\right], \tag{11}$$

6. Morse and Feschbach, op. cit., p. 374.

from which

$$\Delta Z_K = \frac{(\epsilon \eta^{(K)})^2}{(I^{(K)} + \epsilon \eta^{(K)})^2} \left[\frac{Z_{ij} \eta^i \eta^j}{(\eta^K)^2} - Z_K \right] . \tag{12}$$

Because $\delta I^K = \epsilon \eta^K$ appears as a second order factor in (12), the quadric form (6) is said to possess the stationary property. This is further augmented by the fact that it can be shown that the vanishing of the first derivative of an analytic function produces a minimax in its absolute value.[7] However, the bracketed expression in (12) is the difference between the impedance that would be mathematically seen at the driving point, provided the true currents vanished identically, and the true impedance at the driving point. If the η^i are completely arbitrary, this difference may become sufficiently large to produce first order errors in the impedance.

Instead of from the quadric form (2), the impedance may be found from the bilinear form (3),

$$Z_K = \frac{Z_{ij} \bar{I}^i I^j}{|I^K|^2} \tag{13}$$

as before, write,

$$Z_K(\epsilon) = \frac{Z_{ij} (\bar{I}^i + \bar{\epsilon} \bar{\eta}^i)(I^j + \epsilon \eta^j)}{(\bar{I}^K + \bar{\epsilon} \bar{\eta}^K)(I^K + \epsilon \eta^K)} . \tag{14}$$

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7. Appendix A.

A necessary and sufficient condition that $Z_K(\epsilon)$ be an analytic * function of $Z_K(\epsilon)$ is that all order partial derivatives with respect to $\bar{\epsilon}$ vanish, and that the partial derivative with respect to ϵ exists.

Therefore, set,

$$\frac{\partial Z_K(\epsilon)}{\partial \bar{\epsilon}} = 0 . \tag{15}$$

Now,

$$\frac{\partial Z_K(\epsilon)}{\partial \bar{\epsilon}} = \frac{1}{\bar{I}^K + \bar{\epsilon}\bar{\eta}^K}\left[\frac{Z_{ij}\bar{\eta}^i(I^j + \epsilon\eta^j)}{I^K + \epsilon\eta^K} - \eta^{(K)} Z_{(K)}(\epsilon) \right], \tag{16}$$

or,

$$\frac{\partial Z_K(0)}{\partial \bar{\epsilon}} = \frac{1}{\bar{I}^K}\left[\frac{Z_{ij}\bar{\eta}^i I^j}{I^K} - \bar{\eta}^{(K)} Z_{(K)} \right] . \tag{17}$$

As before, inspection reveals that if the voltage law (10) is substituted into (17),

$$\frac{\bar{\eta}^i \delta_i^{(K)} V_{(K)}}{I^{(K)}} = \frac{\bar{\eta}^{(K)} V_{(K)}}{I^{(K)}} = \bar{\eta}^{(K)} Z_{(K)} ,$$

and the derivative vanishes. As a further check, all higher order derivatives are expressable in terms of the first derivative and vanish when subjected to the voltage law. Consequently, again the requirement of analyticity at $\epsilon = 0$ yields the voltage law for the network.

Now, since $Z_K(\epsilon)$ is to be analytic,

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* It is assumed that the imaginary part of the function does not vanish identically.

$$\frac{dZ_K(\epsilon)}{d\epsilon} = \frac{\partial Z_K(\epsilon,\bar{\epsilon})}{\partial \epsilon} = \frac{Z_{ij}(\bar{I}^i + \bar{\epsilon}\bar{\eta}^i)\eta^j}{|I^K + \epsilon\eta^K|^2} - \frac{\eta^{(K)}Z_{(K)}(\epsilon)}{I^K + \epsilon\eta^K} , \quad (18)$$

or,

$$\frac{dZ_K(0)}{d\epsilon} = \frac{\eta^{(K)}}{I^K}\left[\frac{Z_{ij}\bar{I}^i\eta^j}{\bar{I}^K\eta^K} - Z_K\right] . \quad (19)$$

Hence, unless the trivial requirement that η^j be proportional to I^j is imposed, the first derivative is nonvanishing at $\epsilon = 0$. Since the nth derivative, subjected to (10), is,

$$\frac{d^n Z_K(0)}{d\epsilon^n} = n!\left(\frac{-\eta^{(K)}}{I^{(K)} + \epsilon\eta^{(K)}}\right)^{n-1}\frac{dZ_K(0)}{d\epsilon} , \quad (20)$$

the Taylor's expansion about $\epsilon = 0$ becomes,

$$\Delta Z_K = \frac{\delta I^{(K)}}{I^K + \delta I^K}\left[\frac{Z_{ij}\bar{I}^i I^j}{\bar{I}^K\eta^K} - Z_K\right] , \quad (21)$$

in which $\delta I^K = \epsilon\eta^K$, and in which the first term within the bracket is the mutual impedance between the I^i and η^j referred to the driving point.

Although ΔZ_K in (21) apparently is of first order in δI^K, and hence does not possess the stationary property, it may be possible to choose the η^j such that the mutual impedance is of the same order as the self impedance Z_K. Therefore, the error in Z_K may be of the second order even though the bilinear form is not stationary in the ordinary sense. For arbitrary η^i, comparison of (21) with (12) reveals that the variational error from the bilinear form actually may be smaller than the corresponding error from the quadric form.

The preceding discussion of the two algebraic forms frequently

encountered in physical problems, that is, the quadric form and the bilinear form, very aptly illustrates the postulated theory that, optimizing within a complex domain is basically the principle of requiring analyticity, and that its goal is to discover the physical laws governing the subject under consideration.

THE CYLINDRICAL ANTENNA

For a perfectly conducting thin cylindrical antenna, the driving point impedance may be found from [8],

$$Z_0 = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l} R_e\left[\frac{\bar{f}(x)\, f(x')}{|f(0)|^2}\right] G(x-x')\, dx\, dx', \qquad (22)$$

in which,

$$G(x-x') = \left(\frac{\partial^2}{\partial x \partial x'} - K^2\right) \exp(-jKr)/r,$$

x = axial coordinate of antenna,

x' = linear surface coordinate,

$$r = [(x-x')^2 + a^2]^{1/2},$$

a = radius of antenna,

l = half length of antenna,

$2\pi/\lambda = K$ = free space wave number,

$f(x)$ = current distribution function,

$\bar{f}(x)$ = complex conjugate of $f(x)$,

R_e = real part of ... .

8. J. G. Chaney, "A Critical Study of the Circuit Concept", Jour. Appl. Phys., Vol 22, 1429-1436, Dec., 1951.

For thin antennas, it can be shown that using the approximate Green's function, $G(x-x')$, yields the same results as those obtained by using the exact Green's function [9]. That the imaginary part of the current product contributes nothing to Z_0 , may be verified by using the conjugate of the assumed current product, interchanging the variables, and observing the resulting identity of the two integrals produced by the symmetry in the kernel. Equation (22) may be considered as an extension of equation (13), and may be classed as a generalized bilinear form.

Suppose the current function is perturbed while the driving point voltage, V_0 , is kept fixed; that is, write,

$$Z_0(\epsilon,\bar{\epsilon}) = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l}\frac{[\bar{f}(x)+\bar{\epsilon}\,\bar{\varphi}(x)][f(x')+\epsilon\,\varphi(x')]}{[\bar{f}(0)+\bar{\epsilon}\,\bar{\varphi}(0)][f(0)+\epsilon\,\varphi(0)]}G(x-x')\,dx\,dx'. \tag{23}$$

By formally writing the nth partial derivative of Z_0 with respect to $\bar{\epsilon}$,

$$\frac{\partial^n}{\partial\bar{\epsilon}^n}Z_0(\epsilon,\bar{\epsilon}) = n!\left(\frac{-\bar{\varphi}(0)}{\bar{f}(0)+\bar{\epsilon}\,\bar{\varphi}(0)}\right)^{n-1}\frac{\partial Z_0(\epsilon,\bar{\epsilon})}{\partial\bar{\epsilon}}, \tag{24}$$

it is seen that conditions under which the first partial derivative vanishes are sufficient for analyticity. From,

$$\frac{\partial Z_0(\epsilon,\bar{\epsilon})}{\partial\bar{\epsilon}} = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l}\frac{f(x')+\epsilon\varphi(x')}{f(0)+\epsilon\,\varphi(0)}\left[\frac{\bar{\varphi}(x)}{\bar{f}(0)+\bar{\epsilon}\,\bar{\varphi}(0)} - \frac{\bar{\varphi}(0)(\bar{f}(x)+\bar{\epsilon}\,\bar{\varphi}(x))}{(\bar{f}(0)+\bar{\epsilon}\,\bar{\varphi}(0))^2}\right]G\,dx\,dx' \tag{25}$$

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9. C. T. Tai, " A new interpretation of the integral equation formulation of cylindrical antennas", IRE Trans., PGAP, Vol. AP3; pp. 125-127, July, 1955.

write*,

$$\frac{\partial Z_0(\bar{\epsilon},0)}{\partial \bar{\epsilon}} = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l}\frac{f(x')+\epsilon\varphi(x')}{f(0)+\epsilon\varphi(0)}\left[\frac{\bar{\varphi}(x)}{\bar{f}(0)} - \frac{\bar{\varphi}(0)\bar{f}(x)}{\bar{f}(0)^2}\right]G(x-x')\,dx\,dx' = 0. \tag{26}$$

Now, it might be suspected that requiring Z_0 to be analytic at $\epsilon = 0$ would yield a physical law for the antenna. Hence, write,

$$\frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l}\frac{f(x')}{f(0)}\left[\frac{\bar{\varphi}(x)}{\bar{f}(0)} - \frac{\bar{\varphi}(0)\bar{f}(x)}{\bar{f}(0)^2}\right]G(x-x')\,dx\,dx' = 0, \tag{27}$$

or,

$$\int_{-l}^{l}\Theta(x)\left[\frac{\bar{\varphi}(x)}{\bar{\varphi}(0)} - \frac{\bar{f}(x)}{\bar{f}(0)}\right]dx = 0, \tag{28}$$

in which,

$$\Theta(x) = \frac{30}{jK}\int_{-l}^{l}\frac{f(x')}{f(0)}G(x-x')\,dx'. \tag{29}$$

But, from (22),

$$Z_0 = \int_{-l}^{l}\Theta(x)\frac{\bar{f}(x)}{\bar{f}(0)}\,dx. \tag{30}$$

Therefore, from equations (28), (29) and (30),

$$\frac{30}{jK}\int_{-l}^{l}f(x')G(x-x')\,dx' = Z_0 f(0)\,\delta(x), \tag{31}$$

with $\delta(x)$ being the impulse function.

If,

$$V_0 = Z_0 I_0 f(0), \tag{32}$$

it becomes evident that the Kirchoff voltage law has been obtained

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* It should be remembered that ϵ must approach zero along with $\bar{\epsilon}$, and that (26) only is written for explanatory purposes.

for the antenna, namely,

$$\frac{30}{jK}\int_{-l}^{l} f(x')\,G(x'-x)\,dx' = V_0\,\delta(x), \tag{33}$$

For writing the Taylor's expansion, determine,

$$\frac{d^n}{d\epsilon^n} Z_0(\epsilon) = \frac{\partial^n Z_0(\epsilon,\bar{\epsilon})}{\partial \epsilon^n} = n!\left(\frac{-\varphi(0)}{f(0)+\epsilon\,\varphi(0)}\right)^{n-1}\frac{dZ_0(\epsilon)}{d\epsilon}, \tag{34}$$

and find,

$$\Delta Z_0 = \frac{\delta I(0)}{I(0)+\delta I(0)}\left(Z_{12}-Z_0\right), \tag{35}$$

in which, $I(0) = I_0\,f(0)$, $\delta I(0) = \epsilon\,\varphi(0)$,

and

$$Z_{12} = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l}\frac{\bar{I}(x)\ \delta I(x')}{\bar{I}(0)\ \delta I(0)}\,G(x-x')\,dx\,dx'. \tag{36}$$

If the quadric form corresponding to (22) is used, and if analyticity is required by specifying some particular value of the derivative, say the simplest value apparent, zero, examination of the conditions for analyticity about $\epsilon = 0$ shows that precisely the same law (33) is obtained. The corresponding Taylor's expansion is,

$$\Delta Z_0 = \left[\frac{\delta I(0)}{I(0)+\delta I(0)}\right]^2\left(Z_{22}-Z_0\right), \tag{37}$$

with,

$$Z_{22} = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l}\frac{\delta I(x)\ \delta I(x')}{(\delta I(0))^2}\,G(x-x')\,dx\,dx', \tag{38}$$

and as in lumped networks, it does not necessarily follow that an

impedance for the antenna computed by assuming an a priori current distribution along the antenna in the quadric form is any more accurate than one computed by assigning an a priori distribution along the antenna in the bilinear form.

Moreover, if an additional law governing the current distribution were discovered and imposed upon a solution, its accuracy should be improved. With this in mind, return to equation (36).

Suppose $f(x)$ and $\epsilon\,\eta(x)$ were not mutually independent, but were components whose sum were the true current distribution. There would then be no Taylor's expansion, that is to say, the expansion would vanish identically in ϵ.

If the $\mathcal{I}_m[\bar{I}(x)\,\delta I(x')]$ contributed nothing to the integral in (36), the conjugate of $\bar{I}(x)\,\delta I(x')$ could be taken without altering the value of the integral. This would lead to,

$$Z_{12} = Z_{21} = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l}\frac{f(x)\bar{\varphi}(x')}{f(0)\,\bar{\varphi}(0)}\,G(x-x')\,dx\,dx'. \tag{39}$$

Now, the value of $\dfrac{dZ_0(\epsilon)}{d\epsilon}$ used in writing the Taylor's expansion is,

$$\frac{1}{\varphi(0)}\frac{dZ_0(\epsilon)}{d\epsilon} = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l}\frac{\bar{f}(x)}{\varphi(0)\bar{f}(0)}\left[\frac{\varphi(x')}{f(0)+\epsilon\varphi(0)} - \frac{\varphi(0)(f(x')+\epsilon\varphi(x'))}{(f(0)+\epsilon\varphi(0))^2}\right]G\,dx\,dx' \tag{40}$$

If (39) is substituted for (36) in (40), and if $\dfrac{dZ_0(\epsilon)}{d\epsilon}$ is evaluated by means of (31), it is found that the derivative vanishes, and as a consequence, the Taylor's expansion vanishes identically. Physically, it has been found that the true current modes obey the reciprocity theorem. Consequently, the accuracy of an approximation

[illegible]

should be increased by imposing the reciprocity theorem. This imposition leads to ΔZ_0 in (35) subjected to,

$$Z_{12} = Z_{21} = \frac{30}{jK}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} R_e\left[\frac{\bar{I}(x)\ \delta I(x')}{\bar{I}(0)\ \delta I(0)}\right] G(x-x')\,dx\,dx' \tag{41}$$

The question might arise as to whether or not the requirement of analyticity at a nonvanishing value of $\epsilon = \epsilon_0$ would lead to useful physical laws. If so, the resulting laws could be used in computing an approximate value of Z_0 when neither $f(x)$ nor $\varphi(x)$ were the true current distribution.

For simplicity, define,

$$\zeta_{11} = \frac{30}{jK}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \bar{f}(x) f(x') G(x-x')\,dx\,dx'$$
$$\zeta_{22} = \frac{30}{jK}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \bar{\varphi}(x) \varphi(x') G(x-x')\,dx\,dx'$$
$$\zeta_{12} = \frac{30}{jK}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \bar{f}(x) \varphi(x') G(x-x')\,dx\,dx'$$
$$\zeta_{22} = \frac{30}{jK}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \bar{\varphi}(x) f(x') G(x-x')\,dx\,dx', \tag{42}$$

and,

$$Z_{11} = \zeta_{11}/|f_0|^2, \quad Z_{22} = \zeta_{22}/|\varphi_0|^2,$$
$$Z_{12} = \zeta_{12}/\bar{f}_0\varphi_0, \quad Z_{21} = \zeta_{21}/\bar{\varphi}_0 f_0,$$

with,

$$f_0 = f(0), \quad \varphi_0 = \varphi(0).$$

Equation (23) becomes,

$$Z_0(\epsilon, \bar{\epsilon}) = \frac{\zeta_{11} + \epsilon\zeta_{12} + \bar{\epsilon}\zeta_{21} + \epsilon\bar{\epsilon}\zeta_{22}}{(\bar{f}_0 + \bar{\epsilon}\bar{\varphi}_0)(f_0 + \epsilon\varphi_0)}, \tag{43}$$

and equation (26) becomes,

$$(\zeta_{21}+\epsilon\,\zeta_{22})(\bar{f}_0+\bar{\epsilon}\bar{\varphi}_0)-\bar{\varphi}_0(\zeta_{11}+\epsilon\zeta_{12}+\bar{\epsilon}\zeta_{21}+|\epsilon|^2\zeta_{22})=0. \tag{44}$$

Equation (44) is independent of $\bar{\epsilon}$ and its solution is,

$$\epsilon_0=\frac{\bar{\varphi}_0\,\zeta_{11}-\bar{f}_0\,\zeta_{21}}{\bar{f}_0\,\zeta_{22}-\bar{\varphi}_0\,\zeta_{12}}=\frac{f_0}{\varphi_0}\,\frac{Z_{11}-Z_{21}}{Z_{22}-Z_{12}}. \tag{45}$$

Substituting into equation (43), the impedance is found to be,

$$Z_0(\epsilon_0)=\frac{Z_{11}Z_{22}-Z_{12}Z_{21}}{Z_{11}+Z_{22}-Z_{12}-Z_{21}}. \tag{46}$$

The current distribution becomes,

$$I(x)=I_0\left[(Z_{22}-Z_{21})\frac{f(x)}{f(0)}+(Z_{11}-Z_{21})\frac{\varphi(x)}{\varphi(0)}\right]. \tag{47}$$

Now, equations (46) and (47) reveal that the governing law applicable to circuits driven in parallel has been discovered. Thus, write,

$$\begin{aligned} Z_{11}I_1+Z_{12}I_2&=V_0\\ Z_{21}I_1+Z_{22}I_2&=V_0, \end{aligned} \tag{48}$$

and equation (47) becomes,

$$I(x)=\frac{V_0}{Z_{11}Z_{22}-Z_{12}Z_{21}}\left[(Z_{22}-Z_{12})\frac{f(x)}{f(0)}+(Z_{11}-Z_{21})\frac{\varphi(x)}{\varphi(0)}\right]. \tag{49}$$

However, for arbitrary $f(x)$ and $\varphi(x)$, there is no assurance that the $\mathscr{Im}[\bar{f}(x)\,\varphi(x)]$ will contribute nothing to Z_{12} and Z_{21}, and it is likely that $Z_{12}\neq Z_{21}$. To assure symmetry in the subscripts, the reciprocity theorem will be imposed. Hence,

for the final solution, the impedances are,

$$Z_{11} = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l} Re\left[\frac{\bar{f}(x)\, f(x')}{|f(0)|^2}\right] G(x-x')\, dx\, dx', \tag{50}$$

$$Z_{22} = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l} Re\left[\frac{\bar{\varphi}(x)\, \varphi(x')}{|\varphi(0)|^2}\right] G(x-x')\, dx\, dx', \tag{51}$$

$$Z_{12} = Z_{21} = \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l} Re\left[\frac{\bar{f}(x)\varphi(x')}{\bar{f}(0)\varphi(0)}\right] G(x-x')\, dx\, dx'. \tag{52}$$

The imposition of the reciprocity theorem somewhat may be justified algebraically be recalling that it is the impedance function which has been optimized, and by noting that the resulting formula (46) is symmetric in the subscripts. Then, as far as the equivalent circuit equations (48) are concerned Z_{12} and Z_{21} could be interchanged without altering the impedance $Z_0(\epsilon_0)$. However, Z_{12} and Z_{21} would be interchanged in equation (49), resulting in a different current distribution. If the two current distributions were averaged over the entire antenna, a resulting averaging of the corresponding driving point equations (48) would occur. This averaging would not affect the denominator of $Z_0(\epsilon_0)$, but it would affect the numerator. Thus, unless both $f(X)$ and $\varphi(X)$ are real, the imposition of the reciprocity theorem results in a revision of the current distribution along the antenna with a corresponding revision of the driving point impedance. This revision should increase the accuracy of the solution.

The same governing circuit laws can be obtained from the quadric form by requiring analyticity by specifying the existence of a vanishing first derivative. This is the variational solution introduced by

Storer[10] and subsequently used by Tai[11]. However, neither writer pointed out the physical significance of the resulting formulas. Indeed, Storer[12] mistakenly[13] implied that $|Z_0(\epsilon) - Z_0|$ was being made a minimum.

Numerical results, based upon equations (46), (48), and (49), have been presented by the writer[14] for $f(x) = \sin K(l-|x|)$ and $\varphi(x) = \alpha(l-|x|)\, e^{-jK|x|}$. Impedances (50), (51), and (52) were intuitively substituted for the corresponding impedances of the variational method based upon the quadric form.

It should be pointed out that if $\varphi(0) = 0$, Z_{12} and Z_{21} could be referred to any value $x = x_0$ for which $\varphi(x_0) \neq 0$, and the current determined by $\varphi(x)$ becomes parasitically excited.

ARBITRARY NUMBER OF CURRENTS ON ANTENNA

It will be shown that the circuit theory, obtained for the superposition of two current modes along an antenna, can be extended to the superposition of an arbitrary number of current modes.

Let $z^{(i)}$ be a constant internal impedance per unit length. The driving point impedance may be written,

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10. J. E. Storer, "Variational Solution to the Problem of the Symmetrical Antenna", Craft Lab., Harvard Univ. Cambridge, Mass.; Tech. Rept. No. 101, 1950.
11. C. T. Tai, "A Variational Solution to the Problem of Cylindrical Antennas", Stanford Res. Inst., Stanford, Cal., Tech. Rept. No. 12, 1950.
12. Storer, op. cit., p. 7.
13. Appendix A.
14. J. G. Chaney, "A Simple Solution to the Problem of the Cylindrical Antenna", U.S. Naval Postgraduate School, Monterey, Cal., Tech. Rept. No. 15, Jan., 1956.

$$Z_o = Z^{(i)}\int_{-l}^{l}\left|\frac{I(x)}{I(0)}\right|^2 dx + \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l} Re\left[\frac{\bar{I}(x)\, I(x')}{\bar{I}(0)\, I(0)}\right] G(x-x')\, dx\, dx'. \tag{53}$$

Let,

$$I(x) = A^r f_r(x), \quad r = 1, 2, \cdots, n, \tag{54}$$

with the A^r being parameters replacing ϵ . Upon dropping the operator Re and using $a_r = f_r(0)$,

$$Z_o(A^r, \bar{A}^r) = Z^{(i)}\int_{-l}^{l} \frac{\bar{A}^r \bar{f}_r(x)\, A^s f_s(x)}{\bar{a}_r \bar{A}^r a_s A^s}\, dx$$

$$+ \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l} \frac{\bar{A}^r \bar{f}_r(x)\ A^s f_s(x)}{\bar{a}_r \bar{A}^r\ a_s\ A^s}\, G(x-x')\, dx\, dx'. \tag{55}$$

Let,

$$\zeta_{rs} = Z^{(i)}\int_{-l}^{l} \bar{f}_r(x) f_s(x')\, dx + \frac{30}{jK}\int_{-l}^{l}\int_{-l}^{l} \bar{f}_r(x) f_s(x') G(x-x')\, dx\, dx', \tag{56}$$

and

$$Z_{rs} = \zeta_{rs} / \bar{a}_{(r)}\, a_{(s)}\ . \tag{57}$$

Then,

$$Z_o(A^r, \bar{A}^r) = \frac{\zeta_{rs}\, \bar{A}^r A^s}{\bar{a}_r \bar{A}^r a_s A^s}, \quad r, s = 1, \cdots, n. \tag{58}$$

Analyticity will be assured provided, for each r, the partial derivatives with respect to the parameters, $\bar{A}^r$, vanish. Hence,

$$\frac{\zeta_{rs} A^s}{\bar{a}_r \bar{A}^r a_s A^s} - \frac{\bar{a}_r\, \zeta_{st}\, \bar{A}^s A^t}{(\bar{a}_r \bar{A}^r)^2 (A^s a_s)^2} = 0, \tag{59}$$

or,

$$\zeta_{rs} \bar{A}^s \bar{a}_t \bar{A}^t - \bar{a}_r\, \zeta_{su} \bar{A}^s A^u = 0, \tag{60}$$

and $\underline{n}$ equations have been obtained.

Factoring $\bar{A}^t$,

$$\bar{A}^t[\bar{a}_t \zeta_{rs} A^s - \bar{a}_r \zeta_{tu} A^u] = 0. \tag{61}$$

If, for each $\underline{t}$, the coefficients of $\bar{A}^t$ are required to vanish, $\underline{n}^2$ linear homogeneous equations are obtained in the $\underline{n}$ parameters, namely,

$$\bar{a}_t \zeta_{rs} A^s - \bar{a}_r \zeta_{tu} A^u = 0. \tag{62}$$

It will be shown that equations (62) are linearly dependent upon a subset of $(n - 1)$ linearly independent homogeneous equations.

For t = 1,

$$\bar{a}_1 \zeta_{rs} A^s - \bar{a}_r \zeta_{1u} A^u = 0. \tag{63}$$

For $t \neq 1$, multiply equations (63) by $\bar{a}_t$ and equations (62) by $\bar{a}_1$.

After subtracting,

$$\bar{a}_r(\bar{a}_1 \zeta_{tu} A^u - \bar{a}_t \zeta_{1u} A^u) = 0. \tag{64}$$

Thus, for each $\underline{r}$, the subset (63) is obtained.

Factoring $\underline{A^s}$ and dividing by $\bar{a}_1$, dquations (63) may be written as,

$$I^s(Z_{rs} - Z_{1s}) = 0, \tag{65}$$

in which $I^{(s)}$ is defined as $a_{(s)} A^{(s)}$. A necessary and sufficient condition for the solution of equations (65) for the I^s is the vanishing of the determinant $|Z_{rs} - Z_{1s}|$. The vanishing of the determinant may be verified by noting that the first row vanishes identically.

Therefore, equations (65) reduce to

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$(Z_{i1}-Z_{11})I^1+(Z_{ij}-Z_{1j})I^j=0,\ i,j=2,\cdots,n, \tag{66}$$

in which, due to the hypothesis of n distinct current modes, $Z_{ij}\neq Z_{1j}$. Thus, equations (66) constitute (n - 1) linearly independent equations sufficient to solve for the ratios I^j/I^1.

Now, write the n equations,

$$Z_{rs}I^s=V_0,\ r,s=1,\cdots,n. \tag{67}$$

Subtraction of the first equation from each of the other (n - 1) equations results in the system,

$$Z_{1s}I^s=V_0,\ s=1,\cdots,n,$$

$$(Z_{i1}-Z_{11})I^1+(Z_{ij}-Z_{1j})I^j=0,\ i,j=2,\cdots,n. \tag{68}$$

Since equations (66) must yield the ratios I^j/I^1 for any nonvanishing I^1, and since the additional linearly independent equation in (68) simply serves to assign a value to I_0 in terms of a parameter V_0, the ratios may be found by solving the system (67).

Let $V_r=V_0$ for $r=1,\cdots,n$. Then,

$$I^s=Y^{sr}V_r=\frac{V_0}{\Delta}\sum_{r=1}^{n}\Delta^{sr}, \tag{69}$$

or,

$$I^j/I^1=\sum_{r=1}^{n}\Delta^{j(r)}\Big/\sum_{r=1}^{n}\Delta^{1(r)}, \tag{70}$$

in which Δ^{sr} is the cofactor of Z_{rs} in the matrix $\|Z_{rs}\|$, and in which Δ is the determinant $|Z_{rs}|$.

Since $A^j/A^1=a_1/a_j$, the current distribution may be written,

$$I(x)=I_0\sum_{r=1}^{n}\sum_{s=1}^{n}\Delta^{(s)(r)}\frac{f_r(x)}{f_r(0)}. \tag{71}$$

Substituting from equations (70) into equations (58), each term of (58) may be written,

$$\frac{\sum_{t=1}^{n}\sum_{u=1}^{n} \zeta_{(r)(s)}\, \bar{\Delta}^{(r)(t)}\, \Delta^{(s)(u)}}{\bar{a}_{(r)}(\bar{a}_{(s)}) \left| \sum_{r=1}^{n} \sum_{s=1}^{n} \Delta^{(s)(r)} \right|^{2}} \;.$$

Substituting $Z_{(r)(s)}$ for $\zeta_{(r)(s)} / \bar{a}_{(r)} \bar{a}_{(s)}$ and summing over r and s,

$$Z_0 = \frac{\sum_{t=1}^{n}\sum_{u=1}^{n} Z_{rs}\, \bar{\Delta}^{r(t)}\, \Delta^{s(u)}}{\left| \sum_{r=1}^{n} \sum_{s=1}^{n} \Delta^{(s)(r)} \right|^{2}} \;. \tag{72}$$

Since,

$$Z_{rs}\, \Delta^{s(u)} = \Delta\, \delta_{r}^{(u)} \;,$$

equation (72) becomes,

$$Z_0 = \frac{\Delta \sum_{u=1}^{n}\sum_{t=1}^{n} \bar{\Delta}^{(u)(t)}}{\left| \sum_{r=1}^{n} \sum_{s=1}^{n} \Delta^{(s)(r)} \right|^{2}} \;, \tag{73}$$

or finally,

$$Z_0 = \frac{\Delta}{\sum_{r=1}^{n} \sum_{s=1}^{n} \Delta^{(s)(r)}} \;. \tag{74}$$

From equations (71), if the applied voltage is V_0,

$$Z_0 = \frac{V_0}{I(0)} = \frac{V_0}{I_0 \sum_{r=1}^{n} \sum_{s=1}^{n} \Delta^{(s)(r)}} \tag{75}$$

By equating from (74) and (75), it is found that,

$$I_0 = V_0 / \Delta \;.$$

Hence,

$$I(x) = \frac{V_0}{\Delta} \sum_{n=1}^{m} \sum_{s=1}^{m} \Delta^{(s)(n)} \frac{f_{(n)}(x)}{f_{(n)}(0)} . \tag{76}$$

In consequence of equations (74) and (76), the circuit equations (67) may be written directly from lumped network theory and the current distribution and driving point impedance may be determined therefrom.

The governing equations (67) can be derived far more easily by setting up the complex power bilinear form, perturbing the currents, using the analogue of equation (5), and requiring analyticity of the δ_{ns} in A^s.

The circuit equations also can be found from the analogous quadric form impedance function by specifying the existence of a vanishing first derivative. The subsequent algebra is the same as for the bilinear form.

However, for the bilinear form, unless the current modes are natural modes and have adjusted themselves in accordance with the reciprocity theorem, the circuit impedances in equations (67) will be assymetric in the subscripts. Since each current mode should be as nearly as possible, forced to obey all the laws of the true modes, the reciprocity theorem will be imposed. Equations (57) become replaced by,

$$Z_{ns} = z^{(i)} \int_{-\ell}^{\ell} R_e \left[\frac{\bar{f}_n(x) f_s(x)}{\bar{f}_n(0) f_s(0)} \right] dx + \frac{30}{jK} \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} R_e \left[\frac{\bar{f}_n(x) f_s(x')}{\bar{f}_n(0) f_s(0)} \right] G(x-x') \, dx \, dx'. \tag{77}$$

Perhaps it should be pointed out that, with respect to the internal impedance, the effective value of I^S is,

$$I^{s}_{(rs)} = \frac{I^{s}}{2l} \int_{-l}^{l} \mathrm{Re}\left[\frac{\bar{f}_{(r)}(x) f_{(s)}(x)}{\bar{f}_{r}(0) f_{(s)}(0)} \right] dx, \tag{78}$$

which, for r = s, reduces to,

$$I^{r}_{(rs)} = \frac{I^{r}}{2l} \int_{-l}^{l} \left| \frac{f_{(r)}(x)}{f_{(r)}(0)} \right| dx. \tag{79}$$

Hence, the self and mutual internal impedances may be computed in the ordinary manner of coupled circuits and added to the corresponding radiation impedances, provided the bilinear form of the impedance function is used.

COUPLED ANTENNAS

The circuit laws governing an arbitrary number of coupled antennas, arbitrarily oriented with respect to each other, and each having an arbitrary number of current modes, have been set forth by the writer in a recent paper[15]. The resulting circuitry was an intuitive extension of the variational solution by Leavis and Tai[16] of an arbitrary number of modes on an arbitrary number of straight coplanar antennas, each normal to the line of centers. As in the case of previous variational solutions, the authors, Leavis and Tai, did not point out the physical significance of their solution with respect to the generalized circuitry.

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15. J. G. Chaney, "On coupled antennas of unequal sizes", 25 May, 1956, submitted for publication in IRE Trans. PGAP.
16. C. A. Leavis and C. T. Tai, "A method of analyzing coupled antennas of unequal sizes", IRE Trans. PGAP, Vol. AP-4, pp. 128-132, April, 1956.

It is proposed to derive the circuit equations by requiring analyticity in the parameters for the perturbed currents in the bilinear form impedance functions.

The driving point equations for n antennas may be written,

$$Z_{rs}\, I^{(s)}(0) = V_r \ , \quad r, s = 1, \cdots, n, \tag{80}$$

with [17],

$$Z_{rr} = Z_{(r)}^{(i)} \int_{x_{(r)}} Re\left[\frac{\bar{I}(x_{(r)})\ I(x_{(r)})}{\bar{I}^r(0)\ \ I^r(0)}\right] dx_{(r)}$$

$$+ \frac{30}{jK} \int_{x_{(r)}} \int_{x'_{(r)}} Re\left[\frac{\bar{I}(x_{(r)})\, I(x_{(r)})}{\bar{I}^r(0)\ \ I^r(0)}\right] G(x_{(r)}, x'_{(r)})\, dx_r\, dx'_r \ , \tag{81}$$

and for $r \neq s$,

$$Z_{rs} = \frac{30}{jK} \int_{x_{(r)}} \int_{x_{(s)}} Re\left[\frac{\bar{I}(x_{(r)})\, I(x_{(s)})}{\bar{I}^r(0)\ \ I^s(0)}\right] G(x_{(r)}, x_{(s)})\, dx_{(r)}\, dx_{(s)} \tag{82}$$

in which,

$$G(x_{(r)}, x_{(s)}) = \left[\frac{\partial^2}{\partial x_{(r)}\, \partial x_{(s)}} - K^2 \cos(x_{(r)}, x_{(s)})\right] \psi(x_{(r)}, x_{(s)}), \tag{83}$$

$$\psi(x_{(r)}, x_{(s)}) = \exp(-jK R_{(r)(s)}) / R_{(r)(s)} \quad , \tag{84}$$

$R_{(r)(s)}$ is the distance between points,

$X_{(r)}$, $X_{(s)}$ are curvilinear coordinates.

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17. J. G. Chaney, "A critical study of the circuit concept", loc. cit., p. 1433.

Following Leavis and Tai [18], let,

$$\underline{I}^{r}(x_{(r)}) = A^{r}_{p_r} f^{p_r}(x_{(r)}), \; p_r, q_r = m_r, m_r+1, \ldots, n_r, \tag{85}$$

with $m_1 = 1$, $m_{r+1} = n_r + 1$. Thus, for each $\underline{r}$, there are

$m_{r+1} - m_r = n_r - n_{r-1}$ modes, and there are a total of

$n_m = \sum_{r=1}^{m} (n_r - n_{r-1})$ modes altogether.

Define $a^{p_r} = f^{p_r}(0)$. Then,

$$I^{r}(0) = a^{p_r} A^{r}_{p_r}. \tag{86}$$

For Z_{rr} an interchange of variables shows that it is unnecessary to take the real part of the current product. For Z_{rs}, the integration can be performed by dropping the real operator, repeating the integration for the conjugate of the current product, and averaging the two results.

Substituting from (85) into (81) and (82), after dropping the operator $\underline{\mathrm{Re}}$,

$$Z_{(r)(r)} \bar{I}^{(r)}(0) I^{(r)}(0) = Z^{(i)}_{(r)} \int_{x_{(r)}} \bar{A}^{(r)}_{p_r} \bar{f}^{p_r}(x_{(r)}) A^{(r)}_{q_r} f^{q_r}(x_{(r)})\, dx_{(r)}$$
$$+ \frac{30}{jk} \int_{x_{(r)}} \int_{x'_{(r)}} \bar{A}^{(r)}_{p_r} \bar{f}^{p_r}(x_{(r)}) A^{(r)}_{q_r} f^{q_r}(x'_{(r)}) G(x_{(r)} - x'_{(r)})\, dx_{(r)}\, dx'_{(r)}, \tag{87}$$

$$Z_{(r)(s)} \bar{I}^{(r)}(0)\, I^{(s)}(0) =$$
$$\frac{30}{jk} \int_{x_{(r)}} \int_{x_{(s)}} \bar{A}^{(r)}_{p_r} \bar{f}^{p_r}(x_{(r)}) A^{(s)}_{p_s} f^{p_s}(x_{(s)}) G(x_{(r)} - x_{(s)})\, dx_{(r)}\, dx_{(s)}. \tag{88}$$

Define,

$$I^{(r)}_{(p_r)} = a^{(p_r)} A^{(r)}_{(p_r)},$$

$$Z^{p_r q_r}_{(r)(r)} = Z^{(i)}_{(r)} \int_{x_{(r)}} \bar{f}^{p_r}(x_{(r)}) f^{q_r}(x_{(r)}) / \bar{a}^{(p_r)} a^{(q_r)}\, dx_{(r)}$$
$$+ \frac{30}{jk} \int_{x_{(r)}} \int_{x'_{(r)}} \frac{\bar{f}^{p_r}(x_{(r)})\, f^{q_r}(x'_{(r)})}{\bar{a}^{p_r}\, a^{q_r}} G(x'_{(r)} - x_{(r)})\, dx'_{(r)}\, dx_{(r)}$$

18. Leavis and Tai, op. cit., p. 131.

$$Z^{p_r q_s}_{(r)(s)} = \frac{30}{jK} \int_{x_{(r)}} \int_{x_{(s)}} \frac{\bar{f}^{p_r}(x_{(r)}) f^{q_s}(x_{(s)})}{\bar{a}^{(p_r)} \, a^{(q_s)}} G(x_{(r)} - x_{(s)}) \, dx_{(r)} \, dx_{(s)} \, , \tag{90}$$

and,

$$\zeta^{(p_r)(p_s)}_{rs} = Z^{(p_r)(p_s)}_{rs} \bar{a}^{(p_r)} a^{(q_s)} \, . \tag{91}$$

Then, equations (87) and (88) may be expressed as,

$$Z_{(r)(s)} \bar{I}^{(r)}(0) I^{(s)}(0) = Z^{p_r q_s}_{(r)(s)} \bar{I}^{(r)}_{p_r} I^{(s)}_{q_s} \, . \tag{92}$$

Substituting from equations (86), (89) and (91) into equations (92), twice the complex power, $\tilde{W}$, may be obtained by summing over the indices $\underline{r}$ and $\underline{s}$. Hence,

$$2\tilde{W} = Z_{rs} \bar{A}^{r}_{p_r} A^{s}_{p_s} \bar{a}^{p_r} a^{p_s} = \zeta^{p_r p_s}_{rs} \bar{A}^{r}_{p_r} A^{s}_{p_s} \tag{93}$$

The governing circuit laws may be determined by finding the rate of change of twice the complex power with respect to the conjugate currents, $\bar{I}^{r}_{p_r}$, and requiring the antenna self and mutual impedances to be analytic in the $I^{r}_{p_r}$, that is, by requiring,

$$\frac{\partial}{\partial \bar{A}^{(r)}_{(p_r)}} Z_{(r)(s)} = 0 \, . \tag{94}$$

Since equation (93) is an identity,

$$2\tilde{W} = \bar{a}^{p_r} Z_{rs} A^{s}_{p_s} a^{p_s} = \zeta^{p_r q_s}_{rs} A^{s}_{q_s} \, . \tag{95}$$

Substituting from equations (80),

$$\zeta^{(p_r)q_s}_{r\ s} A^{s}_{q_s} = \bar{a}^{(p_r)} V_r , \quad r = 1, \cdots, n. \tag{96}$$

Thus, there are $n_r - m_r - 1$ equal values, V_r, for the partial derivatives of the right member of (92) with respect to $\bar{I}^{r}_{p_r}$. Hence, for each value of p_r, define $V^{p_r}_{r} = V_r$.

Then, upon multiplying and dividing by $a^{(p_s)}$, equations (96) become,

$$Z^{p_r q_s}_{r\ s} I^{s}_{q_s} = V^{p_r}_{r} . \tag{97}$$

Equations (97) constitute a system for a network of n_n meshes, with, for each $r = 1, \cdots, n$, $n_r - m_r + 1$ meshes being driven in parallel. Therefore, the circuit relationship has been established.

Let $\Delta^{sr}_{q_s p_r}$ be the cofactor of $Z^{p_r q_s}_{r\ s}$ in the matrix $\| Z^{p_r q_s}_{r\ s} \|$, and let $\Delta = | Z^{p_r q_s}_{r\ s} |$. Then,

$$Y^{sr}_{q_s p_r} = \frac{1}{\Delta} \Delta^{sr}_{q_s p_r} , \tag{98}$$

and,

$$I^{s}_{q_s} = Y^{sr}_{q_s p_r} V^{p_r}_{r} = \sum_{i=1}^{n_r - m_r} Y^{sr}_{q_s\, m_r + i} V_r , \tag{99}$$

or,

$$A^{s}_{q_s} = \frac{1}{a^{(q_s)}} \sum_{i=1}^{n_r - m_r} Y^{sr}_{q_s\, m_r + i} V_r . \tag{100}$$

Therefore,

$$I^{s}(x_s) = \sum_{i=1}^{n_r - m_r} Y^{sr}_{q_s\, m_r + i} \frac{f^{q_s}(x_{(s)})}{f^{q_s}(0)} V_r , \tag{101}$$

and,

$$I^{s}(0) = \sum_{i=1}^{n_r - m_r} \sum_{j=1}^{n_s - m_s} Y^{sr}_{m_s+j,\, m_r+i} V_r \tag{102}$$

or,

$$Y^{sr} V_r = I^{s}(0) , \tag{103}$$

with,

$$Y^{sr} = \sum_{i=1}^{n_r - m_r} \sum_{j=1}^{n_s - m_s} Y^{sr}_{m_s+j,\, m_r+i} = \frac{1}{\Delta} \sum_{i=1}^{n_r - m_r} \sum_{j=1}^{n_s - m_s} \Delta^{sr}_{m_s+j,\, m_r+i} . \tag{104}$$

Substituting from (103) into (80),

$$(Z_{rs} Y^{st} - \delta^{t}_{r}) V_t = 0 , \tag{105}$$

or, since (105) must hold for a set of nonvanishing V_t ,

$$Z_{rs} Y^{st} - \delta^{t}_{r} = 0 . \tag{106}$$

Let D_{ts} be the cofactor of Y^{st} in $\| Y^{st} \|$ and $D = | Y^{st} |$. Then,

$$Z_{rs} = \frac{1}{D} D_{ts} \delta^{t}_{r} = \frac{1}{D} D_{rs} . \tag{107}$$

Again, the accuracy of a solution should be increased by requiring the circuit impedances to obey the reciprocity theorem. That is, the operator Re should be applied to the current products in equations (90) and (91).

The dropping of the real operator in equations (87) and (88) now must be justified. For algebraic simplicity, spinor[19] notation will be used. That is, a bar will be placed over the indices associated with the conjugate current functions in equations (90) and (91).

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19. Gabriel Kron, loc, cit., pp. 345-349.

Basically, this is simply a means of indicating whether an index is to represent a row or a column in a matrix. Thus, equations (97) may be re-written as,

$$Z^{p_{\bar r}\ q_s}_{\bar r\ \ s} I^{s\,\cdot}_{q_s\,\cdot} = V^{p_r\,\cdot}_{\bar r\,\cdot} \ , \tag{108}$$

in which the dots indicate the missing columns in the singular matrices.

Using the conjugate of the currents in equations (87) and (88), the complex power becomes,

$$2\tilde{W} = z_{r\bar s} A^{r}_{p_r} \bar{A}^{\bar s}_{p_{\bar s}} a^{p_r} \bar{a}^{p_{\bar s}} = \mathcal{Z}^{p_r\ q_{\bar s}}_{r\ \ \bar s} A^{r}_{p_r} \bar{A}^{\bar s}_{p_{\bar s}} \ . \tag{109}$$

Differentiating and imposing analyticity,

$$2 \frac{\partial \tilde{W}}{\partial \bar{A}^{\bar s}_{p_{\bar s}}} = \bar{a}^{p_{\bar s}} Z_{r\bar s} A^{r}_{p_r} a^{p_r} = \mathcal{Z}^{p_r\ q_{\bar s}}_{r\ \ \bar s} A^{r}_{p_r} \ , \tag{110}$$

and hence the resulting equations are,

$$Z^{p_r\ q_{\bar s}}_{r\ \ \bar s} I^{\cdot\, r}_{\cdot\, p_r} = V^{\cdot\, q_{\bar s}}_{\cdot\, \bar s} \ , \tag{111}$$

in which the dots indicate the missing rows in the singular matrices. Since the matrix $\| Z^{p_r\ q_{\bar s}}_{r\ \ \bar s} \|$ is the transpose of the matrix $\| Z^{p_{\bar r}\ q_s}_{\bar r\ \ s} \|$, and since $\| I^{\cdot\, r}_{\cdot\, p_r} \|$ multiplies on the left whilst $\| I^{s\,\cdot}_{q_s\,\cdot} \|$ multiplies on the right, it follows that equations (111) are identically equations (108).

If any assumed current should vanish at the driving point, that is, if some $a^{p_r} = 0$, the corresponding impedances, $Z^{p_r\ q_s}_{r\ \ s}$, could be referred to any point on the antenna where this current is nonvanishing. Since the left member of (95) would be independent of this $\bar{I}^{r}_{p_r}$,

the derivative of the right member would vanish. Thus, instead of (96), for this P_r,

$$Z^{p_r q_s}_{r\ s} I^{s}_{q_s} = 0. \tag{112}$$

In (97), the corresponding $V^{p_r}_{r}$ would be zero. Consequently, any current which vanishes at the driving point becomes parasitically excited and the generalized network theory is still valid provided the corresponding $V^{p_r}_{r}$ is set equal to zero.

CONCLUSION

The examples which have been discussed serve to illustrate the theory that the solution to a physical problem, set up in a complex domain, may be found by requiring analyticity in the variational form. The form need not necessarily possess the stationary property.

In consequence, the generalized circuit theory based upon the bilinear forms which are nonstationary has been extended to cover any configuration of thin wire antennas. The solutions may possibly be more accurate than solutions based upon quadric forms which are stationary.

It is believed that the theory of optimizing by requiring analyticity can be carried over to other fields, and it is hoped that it will become a useful tool in solving physical problems.

APPENDIX A

It will be shown that if $W(z)$ is analytic, setting $\frac{d}{dz}W(z)=0$ yields a minimax in $|W(z)|$.

Let,

$$W(z) = U(x,y) + jV(x,y).$$

A necessary condition that $W(z)$ be analytic is that the Cauchy-Riemann equations are satisfied, that is,

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}. \tag{1}$$

Also, a necessary condition for an extremum in $|W(z)|$ is that,

$$\frac{\partial |W|}{\partial x} = \frac{1}{|W|}\left[U\frac{\partial U}{\partial x} + V\frac{\partial V}{\partial x}\right] = 0$$

$$\frac{\partial |W|}{\partial y} = \frac{1}{|W|}\left[U\frac{\partial U}{\partial y} + V\frac{\partial V}{\partial y}\right] = 0. \tag{2}$$

A necessary condition for equations (2) to hold simultaneously is the vanishing of the Jacobian, that is,

$$\begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial V}{\partial x} \\ \frac{\partial U}{\partial y} & \frac{\partial V}{\partial y} \end{vmatrix} = 0. \tag{3}$$

From (1) and (3),

$$\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 = 0$$

$$\left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = 0, \tag{4}$$

or,

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial V}{\partial y} = 0. \tag{5}$$

For an analytic function, equations (5) are equivalent to setting,

$$\frac{dW}{dz} = 0. \tag{6}$$

Write the second derivatives and subject them to conditions (5),

$$\left.\frac{\partial^2 |W|}{\partial x^2}\right]_c = \frac{1}{|W|}\left[U\frac{\partial^2 U}{\partial x^2} + V\frac{\partial^2 V}{\partial x^2}\right]_c, \tag{7}$$

$$\left.\frac{\partial^2 |W|}{\partial y^2}\right]_c = \frac{1}{|W|}\left[U\frac{\partial^2 U}{\partial y^2} + V\frac{\partial^2 V}{\partial y^2}\right]_c, \tag{8}$$

$$\left.\frac{\partial^2 |W|}{\partial x\,\partial y}\right]_c = \frac{1}{|W|}\left[U\frac{\partial^2 U}{\partial x\,\partial y} + V\frac{\partial^2 V}{\partial x\,\partial y}\right]_c. \tag{9}$$

Substituting from (1) into (9),

$$\left.\frac{\partial^2 |W|}{\partial x\,\partial y}\right]_c = \frac{1}{|W|}\left[U\frac{\partial^2 V}{\partial y^2} - V\frac{\partial^2 U}{\partial x^2}\right]_c. \tag{10}$$

From (1), the LaPlace equations follow, that is,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{11}$$

Substituting from (11) into (3),

$$\left.\frac{\partial^2 |W|}{\partial y^2}\right]_c = -\left.\frac{\partial^2 |W|}{\partial x^2}\right]_c. \tag{12}$$

Therefore,

$$\left.\frac{\partial^2 |W|}{\partial x^2}\right]_c \left.\frac{\partial^2 |W|}{\partial y^2}\right]_c = -\left[\frac{\partial^2 |W|}{\partial x^2}\right]_c^2 < \left[\frac{\partial^2 |W|}{\partial x\,\partial y}\right]_c^2. \tag{13}$$

Inequality (13) is the condition for an extremum to be a minimax [20], that is, $|W|$ is neither a maximum nor a minimum.

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20. E. B. Wilson, "Advanced Calculus", Ginn and Co., New York, N. Y., pp. 114-115, 1912.

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Optimizing by requiring analyticity with examples in antenna theory.